Notes - Polynomials

Objectives

Identify, evaluate, add, and subtract polynomials.

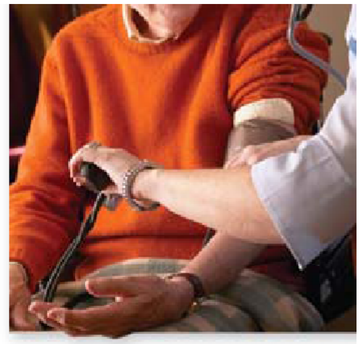
Classify and graph polynomials.

Vocabulary

monomial
polynomial
degree of a monomial
degree of a polynomial
leading coefficient
binomial
trinomial
polynomial function

Who uses this?

Doctors can use polynomials to model blood flow. (See Example 4.)



A **monomial** is a number or a product of numbers and variables with whole number exponents. A **polynomial** is a monomial or a sum or difference of monomials. Each monomial in a polynomial is a term. Because a monomial has only one term, it is the simplest type of polynomial.

Polynomials have no variables in denominators or exponents, no roots or absolute values of variables, and all variables have whole number exponents.

Polynomials: $3x^4$ $\quad 2z^{12} + 9z^3$ $\quad \frac{1}{2}a^7$ $\quad 0.15x^{101}$ $\quad 3t^2 - t^3$

Not polynomials: 3^x $\quad |2b^3 - 6b|$ $\quad \frac{8}{5y^2}$ $\quad \frac{1}{2}\sqrt{x}$ $\quad m^{0.75} - m$

The **degree of a monomial** is the sum of the exponents of the variables.

The **degree of a polynomial** is given by the term with the greatest degree. A polynomial with one variable is in standard form when its terms are written in descending order by degree. So, in standard form, the degree of the first term indicates the degree of the polynomial, and the **leading coefficient** is the coefficient of the first term.

Standard Form

Leading coefficient $\quad$ Degree of polynomial

$$5x^3 + 8x^2 + 3x - 17$$

Degree of term: 3 $\quad$ 2 $\quad$ 1 $\quad$ 0

A polynomial can be classified by its number of terms. A polynomial with two terms is called a **binomial**, and a polynomial with three terms is called a **trinomial**. A polynomial can also be classified by its degree.

Classifying Polynomials by Degree

Name	Degree	Example
Constant	0	-9
Linear	1	$x - 4$
Quadratic	2	$x^2 + 3x - 1$
Cubic	3	$x^3 + 2x^2 + x + 1$
Quartic	4	$2x^4 + x^3 + 3x^2 + 4x - 1$
Quintic	5	$7x^5 + x^4 - x^3 + 3x^2 + 2x - 1$

Add or subtract. Write your answer in standard form.

B $(1 - x^2) - (3x^2 + 2x - 5)$

Add the opposite horizontally.

$(1 - x^2) - (3x^2 + 2x - 5)$

$(-x^2 + 1) + (-3x^2 - 2x + 5)$ *Write in standard form.*

$(-x^2 - 3x^2) + (-2x) + (1 + 5)$ *Group like terms.*

$-4x^2 - 2x + 6$ *Add.*

Remember!
To subtract a polynomial, distribute the negative to all terms.

CHECK IT OUT! Add or subtract. Write your answer in standard form.

3a. $(-36x^2 + 6x - 11) + (6x^2 + 16x^3 - 5)$

3b. $(5x^3 + 12 + 6x^2) - (15x^2 + 3x - 2)$

A **polynomial function** is a function whose rule is a polynomial. In this course, you will study only polynomial functions with one variable.

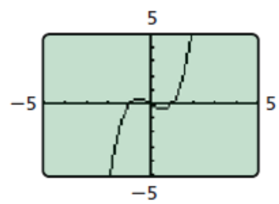
Pay attention to the behavior of the graph!

EXAMPLE 5 Graphing Higher-Degree Polynomials on a Calculator

Graph each polynomial function on a calculator. Describe the graph, and identify the number of real zeros.

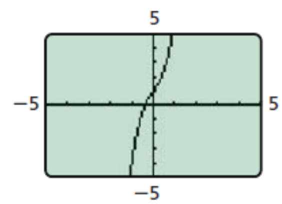
Caution!
Depending on your viewing window, a calculator may not show all of the important features of a graph. Watch out for hidden behavior.

A $f(x) = x^3 - x$



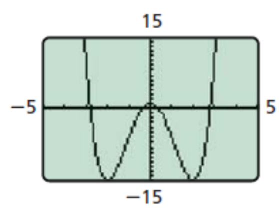
From left to right, the graph increases, decreases slightly, and then increases again. It crosses the x-axis three times, so there appear to be three real zeros.

B $f(x) = 3x^3 + 2x + 1$



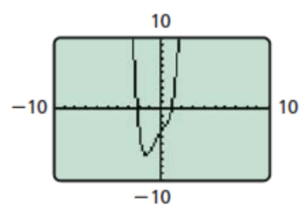
From left to right, the graph increases. It crosses the x-axis once, so there appears to be one real zero.

C $h(x) = x^4 - 8x^2 + 1$



From left to right, the graph alternately decreases and increases, changing direction three times. It crosses the x-axis four times, so there appear to be four real zeros.

D $k(x) = x^4 + x^3 - x^2 + 2x - 3$



From left to right, the graph decreases and then increases. It crosses the x-axis twice, so there appear to be two real zeros.

Multiplying Polynomials

1 Multiplying a Monomial and a Polynomial

Find each product.

A $3x^2(x^3+4)$

$3x^2(x^3+4)$

$3x^2 \cdot x^3 + 3x^2 \cdot 4$ *Distribute.*

$3x^5 + 12x^2$ *Multiply.*

B $ab(a^3+3ab^2-b^3)$

$ab(a^3+3ab^2-b^3)$

$ab(a^3) + ab(3ab^2) + ab(-b^3)$

$a^4b + 3a^2b^3 - ab^4$

CHECK IT OUT! **Find each product.**

1a. $3cd^2(4c^2d - 6cd + 14cd^2)$ **1b.** $x^2y(6y^3 + y^2 - 28y + 30)$

To multiply any two polynomials, use the Distributive Property and multiply each term in the second polynomial by each term in the first.

$$(x+2)(x^2+4x-3)$$

Keep in mind that if one polynomial has m terms and the other has n terms, then the product has mn terms before it is simplified.

In your teacher's opinion, the best method…is the BOX METHOD!

Helpful Hint

When using a table to multiply, the polynomials must be in standard form. Use a zero for any missing terms.

B $(x^2+3x-5)(x^2-x+1)$

Multiply each term of one polynomial by each term of the other. Use a table to organize the products.

	x^2	$-x$	$+1$
x^2	x^4	$-x^3$	$+x^2$
$+3x$	$+3x^3$	$-3x^2$	$+3x$
-5	$-5x^2$	$+5x$	-5

The top left corner is the first term in the product. Combine terms along diagonals to get the middle terms. The bottom right corner is the last term in the product.

$x^4 + (3x^3 - x^3) + (-5x^2 - 3x^2 + x^2) + (5x + 3x) + (-5)$

$x^4 + 2x^3 - 7x^2 + 8x - 5$

CHECK IT OUT! **Find each product.**

2a. $(3b-2c)(3b^2 - bc - 2c^2)$ **2b.** $(x^2 - 4x + 1)(x^2 + 5x - 2)$

Binomial Expansion:

4 Expanding a Power of a Binomial

Find the product.

$(x+y)^3$

$(x+y)(x+y)(x+y)$	*Write in expanded form.*
$(x+y)(x^2+2xy+y^2)$	*Multiply the last two binomial factors.*
$x(x^2)+x(2xy)+x(y^2)+y(x^2)+y(2xy)+y(y^2)$	*Distribute x and then y.*
$x^3+2x^2y+xy^2+x^2y+2xy^2+y^3$	*Multiply.*
$x^3+3x^2y+3xy^2+y^3$	*Combine like terms.*

Notice the coefficients of the variables in the final product of $(x+y)^3$. These coefficients are the numbers from the third row of Pascal's triangle.

Binomial Expansion		Pascal's Triangle (Coefficients)
$(a+b)^0=$	1	1
$(a+b)^1=$	$a+b$	1 1
$(a+b)^2=$	$a^2+2ab+b^2$	1 2 1
$(a+b)^3=$	$a^3+3a^2b+3ab^2+b^3$	1 3 3 1
$(a+b)^4=$	$a^4+4a^3b+6a^2b^2+4ab^3+b^4$	1 4 6 4 1
$(a+b)^5=$	$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$	1 5 10 10 5 1

Each row of Pascal's triangle gives the coefficients of the corresponding binomial expansion. The pattern in the table can be extended to apply to the expansion of any binomial of the form $(a+b)^n$, where n is a whole number.

Binomial Expansion

For a binomial expansion of the form $(a+b)^n$, the following statements are true.

1. There are $n+1$ terms.
2. The coefficients are the numbers from the nth row of Pascal's triangle.
3. The exponent of a is n in the first term, and the exponent decreases by 1 in each successive term.
4. The exponent of b is 0 in the first term, and the exponent increases by 1 in each successive term.
5. The sum of the exponents in any term is n.

Dividing Polynomials

Arithmetic Long Division

Divisor: 12; Dividend: 277; Quotient: 23

$$\begin{array}{r} 23 \\ 12\overline{)277} \\ \underline{24} \\ 37 \\ \underline{36} \\ 1 \end{array}$$

Remainder: 1

Polynomial Long Division

Divisor: $x+2$; Dividend: $2x^2+7x+7$; Quotient: $2x+3$

$$\begin{array}{r} 2x+3 \\ x+2\overline{)2x^2+7x+7} \\ \underline{2x^2+4x} \\ 3x+7 \\ \underline{3x+6} \\ 1 \end{array}$$

Remainder: 1

PS: Ms. Sapp does not like poly. long div!!!

She prefers…

Synthetic division is a shorthand method of dividing a polynomial by a linear binomial by using only the coefficients. For synthetic division to work, the polynomial must be written in standard form, using 0 as a coefficient for any missing terms, and the divisor must be in the form $(x - a)$.

Synthetic Division Method

Divide $(2x^2 + 7x + 9) \div (x + 2)$ by using synthetic division.

WORDS	NUMBERS
Step 1 Write the coefficients of the dividend, 2, 7, and 9. In the upper left corner, write the value of a for the divisor $(x - a)$. So $a = -2$. Copy the first coefficient in the dividend below the horizontal bar.	$\begin{array}{r\|rrr} -2 & 2 & 7 & 9 \\ & \downarrow & & \\ \hline & 2 & & \end{array}$
Step 2 Multiply the first coefficient by the divisor, and write the product under the next coefficient. Add the numbers in the new column.	$\begin{array}{r\|rrr} -2 & 2 & 7 & 9 \\ & & -4 & \\ \hline & 2 & 3 & \end{array}$
Repeat Step 2 until additions have been completed in all columns. Draw a box around the last sum.	$\begin{array}{r\|rrr} -2 & 2 & 7 & 9 \\ & & -4 & -6 \\ \hline & 2 & 3 & \boxed{3} \end{array}$
Step 3 The quotient is represented by the numbers below the horizontal bar. The boxed number is the remainder. The others are the coefficients of the polynomial quotient, in order of decreasing degree.	$= 2x + 3 + \dfrac{3}{x + 2}$

Using Synthetic Division to Divide by a Linear Binomial

Divide by using synthetic division.

A $(4x^2 - 12x + 9) \div \left(x + \frac{1}{2}\right)$

Step 1 Find a. Then write the coefficients and a in the synthetic division format.

$a = -\frac{1}{2}$ *For $\left(x + \frac{1}{2}\right)$, $a = -\frac{1}{2}$.*

$$\begin{array}{r|rrr} -\frac{1}{2} & 4 & -12 & 9 \\ & & & \\ \hline & & & \boxed{} \end{array}$$

Write the coefficients of $4x^2 - 12x + 9$.

Step 2 Bring down the first coefficient. Then multiply and add for each column.

$$\begin{array}{r|rrr} -\frac{1}{2} & 4 & -12 & 9 \\ & & -2 & 7 \\ \hline & 4 & -14 & \boxed{16} \end{array}$$

Draw a box around the remainder, 16.

Step 3 Write the quotient.

$4x - 14 + \dfrac{16}{x + \frac{1}{2}}$ *Write the remainder over the divisor.*

Check Multiply $\left(x + \frac{1}{2}\right)\left(4x - 14 + \dfrac{16}{x + \frac{1}{2}}\right)$.

$$4x\left(x + \frac{1}{2}\right) - 14\left(x + \frac{1}{2}\right) + \frac{16}{x + \frac{1}{2}}\left(x + \frac{1}{2}\right) = 4x^2 - 12x + 9 \checkmark$$

Caution!

Be careful to use the correct a value when doing synthetic division. If the divisor is $(x - a)$, use a. If the divisor is $(x + a)$, use $-a$.

Divide by using synthetic division.

B $(x^4 - 2x^3 + 3x + 1) \div (x - 3)$

Step 1 Find a.

$a = 3$ — *For $(x - 3)$, $a = 3$.*

Step 2 Write the coefficients and a in the synthetic division format.

$$\begin{array}{r|rrrrr} 3 & 1 & -2 & 0 & 3 & 1 \\ \hline & & & & & \end{array}$$

Use 0 for the coefficient of x^2.

Step 3 Bring down the first coefficient. Then multiply and add for each column.

$$\begin{array}{r|rrrrr} 3 & 1 & -2 & 0 & 3 & 1 \\ & & 3 & 3 & 9 & 36 \\ \hline & 1 & 1 & 3 & 12 & \boxed{37} \end{array}$$

Draw a box around the remainder, 37.

Step 4 Write the quotient.

$x^3 + x^2 + 3x + 12 + \frac{37}{x-3}$ — *Write the remainder over the divisor.*

Remainder Theorem

THEOREM	EXAMPLE
If the polynomial function $P(x)$ is divided by $x - a$, then the remainder r is $P(a)$.	Divide $x^3 - 4x^2 + 5x + 1$ by $x - 3$. $\begin{array}{r\|rrrr} 3 & 1 & -4 & 5 & 1 \\ & & 3 & -3 & 6 \\ \hline & 1 & -1 & 2 & \boxed{7} \end{array}$ $P(3) = 7$

3 Using Synthetic Substitution

Use synthetic substitution to evaluate the polynomial for the given value.

A $P(x) = x^3 - 4x^2 + 3x - 5$ for $x = 4$

$$\begin{array}{r|rrrr} 4 & 1 & -4 & 3 & -5 \\ & & 4 & 0 & 12 \\ \hline & 1 & 0 & 3 & \boxed{7} \end{array}$$

Write the coefficients of the dividend. Use $a = 4$.

$P(4) = 7$

Check Substitute 4 for x in $P(x) = x^3 - 4x^2 + 3x - 5$.

$P(4) = 4^3 - 4(4)^2 + 3(4) - 5$

$P(4) = 64 - 64 + 12 - 5$

$P(4) = 7$ ✔

Your remainder is your y-value for the given x-value!

4 *Physics Application*

A Van de Graaff generator is a machine that produces very high voltages by using small, safe levels of electric current. One machine has a current that can be modeled by $I(t) = t + 2$, where $t > 0$ represents time in seconds. The power of the system can be modeled by $P(t) = 0.5t^3 + 6t^2 + 10t$. Write an expression that represents the voltage of the system.

The voltage V is related to current I and power P by the equation $V = \frac{P}{I}$.

$$V(t) = \frac{0.5t^3 + 6t^2 + 10t}{t + 2}$$ *Substitute.*

$$\begin{array}{r|rrrr} -2 & 0.5 & 6 & 10 & 0 \\ & & -1 & -10 & 0 \\ \hline & 0.5 & 5 & 0 & \boxed{0} \end{array}$$ *Use synthetic division.*

The voltage can be represented by $V(t) = 0.5t^2 + 5t$.

CHECK IT OUT! **4.** Write an expression for the length of a rectangle with width $y - 9$ and area $y^2 - 14y + 45$.

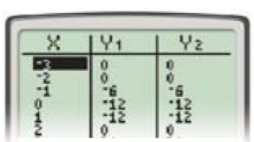
Factoring Polynomials:

Factoring by Grouping

Factor $x^3 + 3x^2 - 4x - 12$.

$(x^3 + 3x^2) + (-4x - 12)$	*Group terms.*
$x^2(x + 3) - 4(x + 3)$	*Factor common monomials from each group.*
$(x + 3)(x^2 - 4)$	*Factor out the common binomial $(x + 3)$.*
$(x + 3)(x + 2)(x - 2)$	*Factor the difference of squares.*

Check Use the table feature of your calculator to compare the original expression and the factored form.



The table shows that the original function and the factored form have the same function values. ✔

Know it! Note

Factoring the Sum and the Difference of Two Cubes

METHOD	ALGEBRA
Sum of two cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of two cubes	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

EXAMPLE 3 Factoring the Sum or Difference of Two Cubes

Factor each expression.

A $5x^4 + 40x$

$5x(x^3 + 8)$ — *Factor out the GCF, 5x.*

$5x(x^3 + 2^3)$ — *Rewrite as the sum of cubes.*

$5x(x + 2)(x^2 - x \cdot 2 + 2^2)$ — *Use the rule $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.*

$5x(x + 2)(x^2 - 2x + 4)$

B $8y^3 - 27$

$(2y)^3 - 3^3$ — *Rewrite as the difference of cubes.*

$(2y - 3)\left[(2y)^2 + 2y \cdot 3 + 3^2\right]$ — *Use the rule $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.*

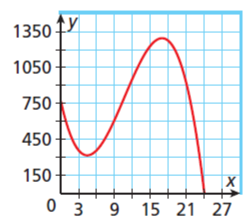
$(2y - 3)(4y^2 + 6y + 9)$

Remember!

GCF stands for "greatest common factor." Always factor out the GCF before using other methods.

Ecology Application

The population of an endangered species of bird in the years since 1990 can be modeled by the function $P(x) = -x^3 + 32x^2 - 224x + 768$. Identify the year that the bird will become extinct if the model is accurate and no protective measures are taken. Use the graph to factor $P(x)$.



Because $P(x)$ represents the population, the real zero of $P(x)$ represents a population of zero, meaning extinction. $P(x)$ has only one real zero at $x = 24$, which corresponds to the year 2014.

If the model is accurate, the bird will become extinct in 2014.

The corresponding factor is $(x - 24)$.

$$\begin{array}{r|rrrr} 24 & -1 & 32 & -224 & 768 \\ & & -24 & 192 & -768 \\ \hline & -1 & 8 & -32 & 0 \end{array}$$

Use synthetic division to factor the polynomial.

$P(x) = (x - 24)(-x^2 + 8x - 32)$ — *Write P(x) as a product.*

$P(x) = -(x - 24)(x^2 - 8x + 32)$ — *Factor out −1 from the quadratic.*

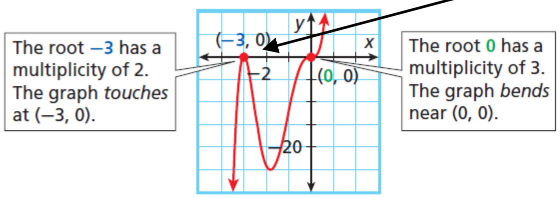
Finding Real Roots:

To find the real roots of a polynomial equation, set every factor equal to zero and solve for x. Your solution is where the function crosses the x-axis (every x-value when y=0).

Sometimes a polynomial equation has a factor that appears more than once. This creates a *multiple root*. In Example 1A, $3x^5 + 18x^4 + 27x^3 = 0$ has two multiple roots, 0 and -3. For example, the root 0 is a factor three times because $3x^3 = 0$.

The **multiplicity** of root r is the number of times that $x - r$ is a factor of $P(x)$. When a real root has even multiplicity, the graph of $y = P(x)$ touches the x-axis but does not cross it. When a real root has odd multiplicity greater than 1, the graph "bends" as it crosses the x-axis.

We refer to this as a "bounce" since the function does not actually cross the x-axis, it just "bounces" off of it.



You cannot always determine the multiplicity of a root from a graph. It is easiest to determine multiplicity when the polynomial is in factored form.

Identifying Multiplicity

Identify the roots of each equation. State the multiplicity of each root.

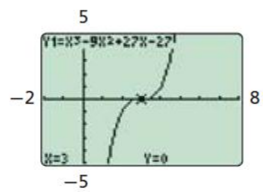
A $x^3 - 9x^2 + 27x - 27 = 0$

$x^3 - 9x^2 + 27x - 27 = (x - 3)(x - 3)(x - 3)$

$x - 3$ is a factor three times.

The root 3 has a multiplicity of 3.

Check Use a graph. A calculator graph shows a bend near $(3, 0)$. ✔



B $-2x^3 - 12x^2 + 30x + 200 = 0$

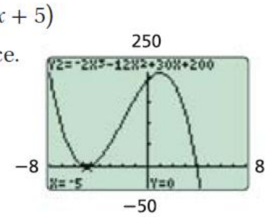
$-2x^3 - 12x^2 + 30x + 200 = -2(x - 4)(x + 5)(x + 5)$

$x - 4$ is a factor once, and $x + 5$ is a factor twice.

The root 4 has a multiplicity of 1.

The root -5 has a multiplicity of 2.

Check Use a graph. The graph crosses at $(4, 0)$ and touches at $(-5, 0)$. ✔



Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra

Every polynomial function of degree $n \geq 1$ has at least one zero, where a zero may be a complex number.

Corollary: Every polynomial function of degree $n \geq 1$ has exactly n zeros, including multiplicities.

Using this theorem, you can write any polynomial function in factored form.

To find all roots of a polynomial equation, you can use a combination of the Rational Root Theorem, the Irrational Root Theorem, and methods for finding complex roots, such as the quadratic formula.

The real numbers are a subset of the complex numbers, so a real number a can be thought of as the complex number $a + 0i$. But here the term *complex root* will only refer to a root of the form $a + bi$, where $b \neq 0$. Complex roots, like irrational roots, come in conjugate pairs. Recall from Chapter 5 that the complex conjugate of $a + bi$ is $a - bi$.

Complex Conjugate Root Theorem

If $\boldsymbol{a + bi}$ is a root of a polynomial equation with real-number coefficients, then $\boldsymbol{a - bi}$ is also a root.

Writing a Polynomial Function with Complex Zeros

Write the simplest polynomial function with zeros $1 + i$, $\sqrt{2}$, and -3.

Step 1 Identify all roots.

By the Irrational Root Theorem and the Complex Conjugate Root Theorem, the irrational roots and complex roots come in conjugate pairs. There are five roots: $1 + i$, $1 - i$, $\sqrt{2}$, $-\sqrt{2}$, and -3. The polynomial must have degree 5.

Step 2 Write the equation in factored form.

$$P(x) = \left[x - (1 + i)\right]\left[x - (1 - i)\right]\left(x - \sqrt{2}\right)\left[x - \left(-\sqrt{2}\right)\right]\left[x - (-3)\right]$$

Step 3 Multiply.

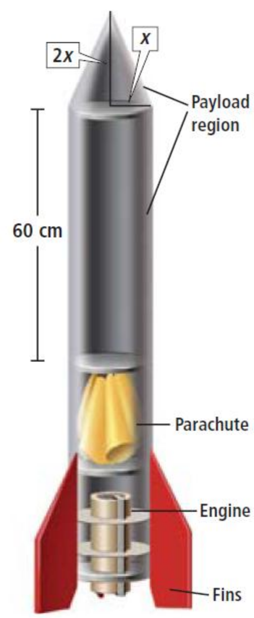
$$P(x) = \left(x^2 - 2x + 2\right)\left(x^2 - 2\right)(x + 3)$$

$$= \left(x^4 - 2x^3 + 4x - 4\right)(x + 3)$$

$$P(x) = x^5 + x^4 - 6x^3 + 4x^2 + 8x - 12$$

Problem-Solving Application

An engineering class is designing model rockets for a competition. The body of the rocket must be cylindrical with a cone-shaped top. The cylinder part must be 60 cm tall, and the height of the cone must be twice the radius. The volume of the payload region must be 558π cm^3 in order to hold the cargo. Find the radius of the rocket.



1 Understand the Problem

The cylinder and the cone have the same radius, x. The answer will be the value of x.

List the important information:

- The cylinder is 60 cm tall.
- The height of the cone part is twice the radius, $2x$.
- The volume of the payload region is 558π cm^3.

2 Make a Plan

Write an equation to represent the volume of the body of the rocket.

$V = V_{\text{cone}} + V_{\text{cylinder}}$

$V(x) = \frac{2}{3}\pi x^3 + 60\pi x^2$ *$V_{cone} = \frac{1}{3}\pi x^2 h$ and $V_{cylinder} = \pi x^2 h$*

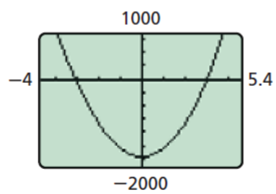
Set the volume equal to 558π.

$\frac{2}{3}\pi x^3 + 60\pi x^2 = 558\pi$

3 Solve

$\frac{2}{3}\pi x^3 + 60\pi x^2 - 558\pi = 0$ *Write in standard form.*

$\frac{2}{3}x^3 + 60x^2 - 558 = 0$ *Divide both sides by π.*



The graph indicates a possible positive root of 3. Use synthetic division to verify that 3 is a root, and write the equation as $(x - 3)\left(\frac{2}{3}x^2 + 62x + 186\right) = 0$. By the quadratic formula, you can find that -3.1 and -89.9 are approximate roots of $\frac{2}{3}x^2 + 62x + 186 = 0$. The radius must be a positive number, so the radius of the rocket is 3 cm.

$$\begin{array}{r|rrrr} 3 & \frac{2}{3} & 60 & 0 & -558 \\ & & 2 & 186 & 550 \\ \hline & \frac{2}{3} & 62 & 186 & \boxed{0} \end{array}$$

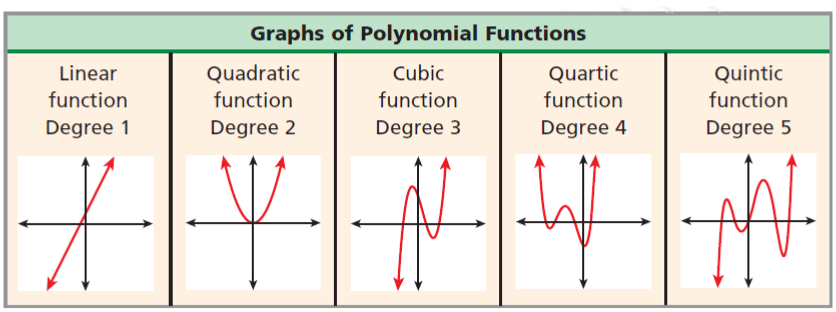
4 Look Back

Substitute 3 cm into the original equation for the volume of the rocket.

$V(3) = \frac{2}{3}\pi(3)^3 + 60\pi(3)^2$

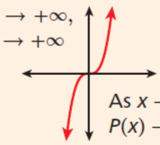
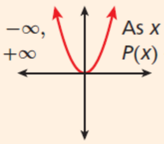
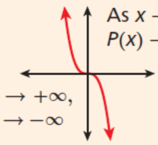
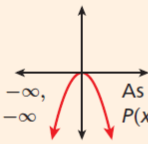
$V(3) = 558\pi$ ✔

Graphs of Polynomial Functions and End Behavior



End behavior is a description of the values of the function as x approaches positive infinity ($x \to +\infty$) or negative infinity ($x \to -\infty$). The degree and leading coefficient of a polynomial function determine its end behavior. It is helpful when you are graphing a polynomial function to know about the end behavior of the function.

Polynomial End Behavior

$P(x)$ has...	Odd Degree	Even Degree
Leading coefficient $a > 0$	As $x \to +\infty$, $P(x) \to +\infty$  As $x \to -\infty$, $P(x) \to -\infty$	As $x \to -\infty$, $P(x) \to +\infty$  As $x \to +\infty$, $P(x) \to +\infty$
Leading coefficient $a < 0$	As $x \to -\infty$, $P(x) \to +\infty$  As $x \to +\infty$, $P(x) \to -\infty$	As $x \to -\infty$, $P(x) \to -\infty$  As $x \to +\infty$, $P(x) \to -\infty$

EXAMPLE 1 **Determining End Behavior of Polynomial Functions**

Helpful Hint

Both the leading coefficient and the degree of the polynomial are contained in the term of greatest degree. When determining end behavior, you can ignore all other terms.

Identify the leading coefficient, degree, and end behavior.

A $P(x) = -4x^3 - 3x^2 + 5x + 6$

The leading coefficient is -4, which is negative.

The degree is 3, which is odd.

As $x \to -\infty$, $P(x) \to +\infty$, and as $x \to +\infty$, $P(x) \to -\infty$.

B $R(x) = x^6 - 7x^5 + x^3 - 2$

The leading coefficient is 1, which is positive.

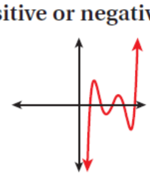
The degree is 6, which is even.

As $x \to -\infty$, $P(x) \to +\infty$, and as $x \to +\infty$, $P(x) \to +\infty$.

Using Graphs to Analyze Polynomial Functions

Identify whether the function graphed has an odd or even degree and a positive or negative leading coefficient.

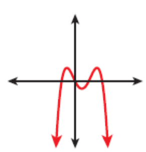
A



As $x \to -\infty$, $P(x) \to -\infty$, and as $x \to +\infty$, $P(x) \to +\infty$.

$P(x)$ is of odd degree with a positive leading coefficient.

B



As $x \to -\infty$, $P(x) \to -\infty$, and as $x \to +\infty$, $P(x) \to -\infty$.

$P(x)$ is of even degree with a negative leading coefficient.

MEMORIZE THIS!!!!

Steps for Graphing a Polynomial Function
1. Find the real zeros and y-intercept of the function.
2. Plot the x- and y-intercepts.
3. Make a table for several x-values that lie between the real zeros.
4. Plot the points from your table.
5. Determine the end behavior of the graph.
6. Sketch the graph.

Graphing Polynomial Functions

Graph the function.

$f(x) = x^3 + 3x^2 - 6x - 8$

Step 1 Identify the possible rational roots by using the Rational Root Theorem.

$\pm1, \pm2, \pm4, \pm8$ *$p = -8$ and $q = 1$*

Step 2 Test possible rational zeros until a zero is identified.

Test $x = 1$.

$$\begin{array}{r|rrrr} 1 & 1 & 3 & -6 & -8 \\ & & 1 & 4 & -2 \\ \hline & 1 & 4 & -2 & \boxed{-10} \end{array}$$

Test $x = -1$.

$$\begin{array}{r|rrrr} -1 & 1 & 3 & -6 & -8 \\ & & -1 & -2 & 8 \\ \hline & 1 & 2 & -8 & \boxed{0} \end{array}$$

$x = -1$ is a zero, and $f(x) = (x + 1)(x^2 + 2x - 8)$.

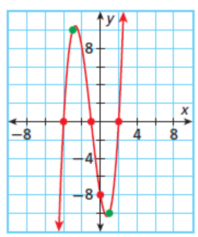
Step 3 Factor: $f(x) = (x + 1)(x - 2)(x + 4)$.
The zeros are -1, 2, and -4.

Step 4 Plot other points as guidelines.
$f(0) = -8$, so the y-intercept is -8.

Plot points between the zeros. Choose $x = -3$ and $x = 1$ for simple calculations.
$f(-3) = 10$ and $f(1) = -10$

Step 5 Identify end behavior.
The degree is odd and the leading coefficient is positive so as $x \to -\infty$, $P(x) \to -\infty$, and as $x \to +\infty$, $P(x) \to +\infty$.

Step 6 Sketch the graph of $f(x) = x^3 + 3x^2 - 6x - 8$ by using all of the information about $f(x)$.



Local Minima and Maxima:

A **turning point** is where a graph changes from increasing to decreasing or from decreasing to increasing. A turning point corresponds to a *local maximum* or *minimum*.

Local Maxima and Minima

For a function $f(x)$, $f(a)$ is a **local maximum** if there is an interval around a such that $f(x) < f(a)$ for every x-value in the interval except a.

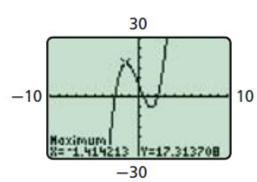
For a function $f(x)$, $f(a)$ is a **local minimum** if there is an interval around a such that $f(x) > f(a)$ for every x-value in the interval except a.

A polynomial function of degree n has at most $n - 1$ turning points and at most n x-intercepts. If the function has n distinct real roots, then it has exactly $n - 1$ turning points and exactly n x-intercepts. You can use a graphing calculator to graph and estimate maximum and minimum values.

EXAMPLE 4 **Determine Maxima and Minima with a Calculator**

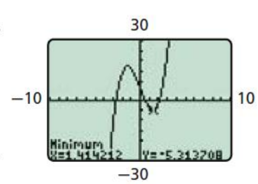
Graph $g(x) = 2x^3 - 12x + 6$ on a calculator, and estimate the local maxima and minima.

Step 1 Graph.
The graph appears to have one local maximum and one local minimum.



Step 2 Find the maximum.
Press 2nd TRACE (CALC) to access the **CALC** menu. Choose **4:maximum**.
The local maximum is approximately 17.3137.

Step 3 Find the minimum.
Press 2nd TRACE (CALC) to access the **CALC** menu. Choose **3:mininum**.
The local minimum is approximately -5.3137.



Reading Math

Maxima is the plural form of *maximum*. *Minima* is the plural form of *minimum*.

Transforming Polynomial Functions:

Transformations of $f(x)$		
Transformation	**$f(x)$ Notation**	**Examples**
Vertical translation	$f(x) + k$	$g(x) = x^3 + 3$ 3 units up $g(x) = x^3 - 4$ 4 units down
Horizontal translation	$f(x - h)$	$g(x) = (x - 2)^3$ 2 units right $g(x) = (x + 1)^3$ 1 unit left
Vertical stretch/compression	$af(x)$	$g(x) = 6x^3$ stretch by 6 $g(x) = \frac{1}{2}x^3$ compression by $\frac{1}{2}$
Horizontal stretch/compression	$f\left(\frac{1}{b}x\right)$	$g(x) = \left(\frac{1}{5}x\right)^3$ stretch by 5 $g(x) = (3x)^3$ compression by $\frac{1}{3}$
Reflection	$-f(x)$ $f(-x)$	$g(x) = -x^3$ across x-axis $g(x) = (-x)^3$ across y-axis

Translations:

Translating a Polynomial Function

For $f(x) = x^3 + 4$, write the rule for each function and sketch its graph.

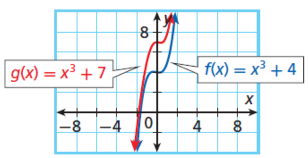
A $g(x) = f(x) + 3$

$g(x) = (x^3 + 4) + 3$

$g(x) = x^3 + 7$

To graph $g(x) = f(x) + 3$, translate the graph of $f(x)$ 3 units up.

This is a vertical translation.



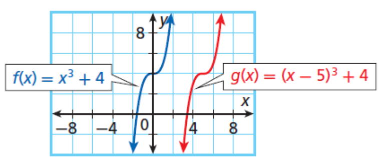
B $g(x) = f(x - 5)$

$g(x) = (x - 5)^3 + 4$

$g(x) = (x - 5)^3 + 4$

To graph $g(x) = f(x - 5)$, translate the graph of $f(x)$ 5 units right.

This is a horizontal translation.



Reflections:

Reflecting Polynomial Functions

Let $f(x) = x^3 - 7x^2 + 6x - 5$. Write a function g that performs each transformation.

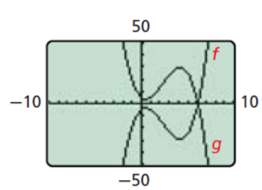
A **Reflect $f(x)$ across the x-axis.**

$g(x) = -f(x)$

$g(x) = -(x^3 - 7x^2 + 6x - 5)$

$g(x) = -x^3 + 7x^2 - 6x + 5$

Check Graph both functions. The graph appears to be a reflection. ✔



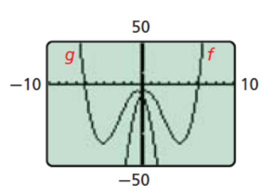
B **Reflect $f(x)$ across the y-axis.**

$g(x) = f(-x)$

$g(x) = (-x)^3 - 7(-x)^2 + 6(-x) - 5$

$g(x) = -x^3 - 7x^2 - 6x - 5$

Check Graph both functions. The graph appears to be a reflection. ✔



Dilations:

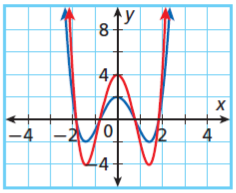
Compressing and Stretching Polynomial Functions

Let $f(x) = x^4 - 4x^2 + 2$. Graph f and g on the same coordinate plane. Describe g as a transformation of f.

A $g(x) = 2f(x)$

$g(x) = 2(x^4 - 4x^2 + 2)$

$g(x) = 2x^4 - 8x^2 + 4$

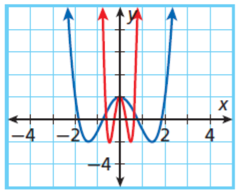


$g(x)$ is a vertical stretch of $f(x)$.

B $g(x) = f(3x)$

$g(x) = (3x)^4 - 4(3x)^2 + 2$

$g(x) = 81x^4 - 36x^2 + 2$



$g(x)$ is a horizontal compression of $f(x)$.

Combining it all:

Combining Transformations

Write a function that transforms $f(x) = 3x^3 + 6$ in each of the following ways. Support your solution by using a graphing calculator.

A **Stretch vertically by a factor of 2, and shift 3 units left.**

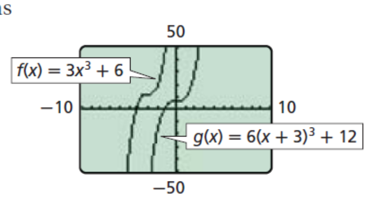
A vertical stretch is represented by $af(x)$, and a horizontal shift is represented by $f(x - h)$. Combining the two transformations gives $g(x) = af(x - h)$.

Substitute 2 for a and 3 for h.

$g(x) = 2f(x + 3)$

$g(x) = 2(3(x + 3)^3 + 6)$

$g(x) = 6(x + 3)^3 + 12$



B **Reflect across the x-axis and shift 3 units up.**

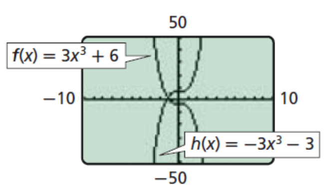
A reflection across the x-axis is represented by $-f(x)$, and a vertical shift is represented by $f(x) + k$. Combining the two transformations gives $h(x) = -f(x) + k$.

Substitute 3 for k.

$h(x) = -f(x) + 3$

$h(x) = -(3x^3 + 6) + 3$

$h(x) = -3x^3 - 3$



Curve Fitting With Polynomials – Essential Question: Given a set of data, how can I determine which polynomial will "fit" the data best?

Finite Differences of Polynomials		
Function Type	**Degree**	**Constant Finite Differences**
Linear	1	First
Quadratic	2	Second
Cubic	3	Third
Quartic	4	Fourth
Quintic	5	Fifth

Using Finite Differences to Determine Degree

Use finite differences to determine the degree of the polynomial that best describes the data.

A

x	-2	-1	0	1	2	3
y	-10	-4	-1.4	0	2.4	8

The x-values increase by a constant 1. Find the differences of the y-values.

y	-10	-4	-1.4	0	2.4	8

First differences:	6	2.6	1.4	2.4	5.6	Not constant
Second differences:	-3.4	-1.2	1	3.2		Not constant
Third differences:	2.2	2.2	2.2			Constant

The third differences are constant. A cubic polynomial best describes the data.

Using Finite Differences to Write a Function

The table below shows the population of a city from 1950 to 2000. Write a polynomial function for the data.

Year	1950	1960	1970	1980	1990	2000
Population (thousands)	2853	4011	5065	6720	9704	14,759

Step 1 Find the finite differences of the y-values.

Let x represent the number of years since 1950. The years increase by a constant amount of 10. The populations are the y-values.

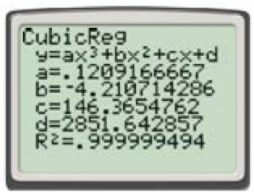
First differences:	1158	1054	1655	2984	5055	
Second differences:	-104	601	1329	2071		
Third differences:	705	728	742			Close

Step 2 Determine the degree of the polynomial.

Because the third differences are relatively close, a cubic function should be a good model.

Step 3 Use the cubic regression feature on your calculator.

$f(x) \approx 0.12x^3 - 4.21x^2 + 146.37x + 2851.64$



Using the regression function on your calculator:

Often, real-world data can be too irregular for you to use finite differences or find a polynomial function that fits perfectly. In these situations, you can use the regression feature of your graphing calculator. Remember that the closer the R^2-value is to 1, the better the function fits the data.

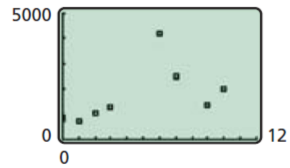
3 *Finance Application*

The table shows the opening value of a stock index on the first day of trading in various years. Use a polynomial model to estimate the value on the first day of trading in 2002.

Year	Price ($)	Year	Price ($)
1994	774	2000	4186
1995	751	2001	2474
1996	1053	2003	1347
1997	1293	2004	2011

Step 1 Choose the degree of the polynomial model.

Let x represent the number of years since 1994. Make a scatter plot of the data.



The function appears to be cubic or quartic. Use the regression feature to check the R^2-values.

cubic: $R^2 \approx 0.6279$ quartic: $R^2 \approx 0.8432$

The quartic function is a more appropriate choice.

Step 2 Write the polynomial model.

The data can be modeled by
$f(x) = 9.27x^4 - 191.56x^3 + 1168.22x^2 - 1702.58x + 999.60$

Step 3 Find the value of the model corresponding to 2002.

2002 is 8 years after 1994. Substitute 8 for x in the quartic model.

$f(x) = 9.27(8)^4 - 191.56(8)^3 + 1168.22(8)^2 - 1702.58(8) + 999.60 \approx$
2036.24

Based on the model, the opening value was about $2036.24 in 2002.